Re-exam Mathematical Physics 2018, Prof. Dr. G. Palasantzas
10 points free
Total points to obtain 100

<u>Problem 1</u> (15 points)

Consider the series: $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\sqrt{n}} x^n$; For which values of x is the series convergent?

<u>Problem 2</u> (15 points)

Find the value of c if

$$\sum_{n=2}^{\infty} (1 + c)^{-n} = 2$$

<u>Problem 3</u> (15 points)

If a, b, and c are all positive constants and $y(x)$ is a solution of the differential equation $ay'' + by' + cy = 0$, show that $\lim_{x \to \infty} y(x) = 0$.

Problem 4 (10 points)

(a: 5 points)
(b: 5 points)

Let L be a nonzero real number.

(a) Show that the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ has only the trivial solution $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) For the case $\lambda > 0$, find the values of λ for which this problem has a nontrivial solution and give the corresponding solution.

Problem 5 (15 points)

Assume a function *f(x)* to have Fourier Transform: $F(k) = \int_{-\infty}^{+\infty} f(x) e^{-i2\pi kx} dx$

Consider also the Fourier Transform definition of the Dirac Delta function: $\delta(k) = \int_{-\infty}^{+\infty} e^{-i2\pi kx} dx$

Derive the Fourier Transform of : (a: 5 points) $f(x) = \cos[4\pi k_o x]$, (b: 10 points) $f(x) = \cos^2[4\pi k_o x]$

Problem 6 (20 points)

Consider the boundary value problem for the one-dimensional heat equation for a bar with the zero-temperature ends:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \quad t > 0, \quad 0 < x < L,$$
$$u(x,0) = f(x),$$
$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0.$$

u(x,t): Temperature

(<u>a: 10 points</u>) Show that the general solution for *u(x,t)* is given by:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x \ , \ \lambda_n = \frac{cn\pi}{L}$$

(<u>b: 10 points</u>) Derive the solution *u(x,t)* for the case *f(x) = 20 sin(πx/L) + 50sin(3 πx/L)*

Problem 1

If $a_n = \dfrac{(-3)^n x^n}{n^{3/2}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{(-3)^n x^n}\right| = \lim_{n\to\infty}\left|-3x\left(\frac{n}{n+1}\right)^{3/2}\right| = 3|x|\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^{3/2}$$

$$= 3|x|(1) = 3|x|$$

By the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{(-3)^n}{n\sqrt{n}}x^n$ converges when $3|x|<1 \iff |x|<\frac{1}{3}$, so $R=\frac{1}{3}$. When $x=\frac{1}{3}$, the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$ converges by the Alternating Series Test. When $x=-\frac{1}{3}$, the series $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is a convergent p-series $\left(p=\frac{3}{2}>1\right)$. Thus, the interval of convergence is $\left[-\frac{1}{3},\frac{1}{3}\right]$.

Problem 2

$\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a=(1+c)^{-2}$ and $r=(1+c)^{-1}$, so the series converges when

$\left|(1+c)^{-1}\right|<1 \quad \Leftrightarrow \quad |1+c|>1 \quad \Leftrightarrow \quad 1+c>1$ or $1+c<-1 \quad \Leftrightarrow \quad c>0$ or $c<-2$. We calculate the sum of the

series and set it equal to 2: $\dfrac{(1+c)^{-2}}{1-(1+c)^{-1}}=2 \quad \Leftrightarrow \quad \left(\dfrac{1}{1+c}\right)^2=2-2\left(\dfrac{1}{1+c}\right) \quad \Leftrightarrow \quad 1=2(1+c)^2-2(1+c) \quad \Leftrightarrow$

$2c^2+2c-1=0 \quad \Leftrightarrow \quad c=\frac{-2\pm\sqrt{12}}{4}=\frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2<\frac{-\sqrt{3}-1}{2}<0$.

So $c=\frac{\sqrt{3}-1}{2}$.

Problem 3

The auxiliary equation is $ar^2+br+c=0$.

If $b^2-4ac>0$, then any solution is of the form

$$y(x)=c_1e^{r_1x}+c_2e^{r_2x} \text{ where } r_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } r_2=\frac{-b-\sqrt{b^2-4ac}}{2a} .$$

But a , b , and c are all positive so both r_1 and r_2 are negative and $y(x)=0$

Here r1<0 is negative because $-b-(b^2-4ac)^{1/2}<0$ and a>0; r2 <0 because $b>(b^2-4ac)^{1/2}$ or $b^2>b^2-4ac$ because ac>0 and a>0

If $b^2-4ac=0$, then any solution is of the form

$y(x)=c_1e^{rx}+c_2xe^{rx}$ where $r=-b/(2a)<0$ since a , b are positive. Hence $y(x)=0$.

if $b^2-4ac<0$

then any solution is of the form $y(x)=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ where $\alpha=-b/(2a)<0$ since a and b are positive. Thus $y(x)=0$.

Problem 4

(a) *Case 1* ($\lambda = 0$): $y'' + \lambda y = 0 \quad \Rightarrow \quad y'' = 0$ which has an auxiliary equation $r^2 = 0 \quad \Rightarrow \quad r = 0 \quad \Rightarrow \quad y = c_1 + c_2 x$

where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0$. Thus $y = 0$.

Case 2 ($\lambda < 0$): $y'' + \lambda y = 0$ has auxiliary equation $r^2 = -\lambda \quad \Rightarrow \quad r = \pm\sqrt{-\lambda}$ [distinct and real since $\lambda < 0$] $\Rightarrow$

$y = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x}$ where $y(0) = 0$ and $y(L) = 0$. Thus $0 = y(0) = c_1 + c_2$ $(*)$ and

$0 = y(L) = c_1 e^{\sqrt{-\lambda} L} + c_2 e^{-\sqrt{-\lambda} L}$ $(\dagger)$.

Multiplying $(*)$ by $e^{\sqrt{-\lambda} L}$ and subtracting $(\dagger)$ gives $c_2\left(e^{\sqrt{-\lambda} L} - e^{-\sqrt{-\lambda} L}\right) = 0 \quad \Rightarrow \quad c_2 = 0$ and thus $c_1 = 0$ from $(*)$.

Thus $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) $y'' + \lambda y = 0$ has an auxiliary equation $r^2 + \lambda = 0 \quad \Rightarrow \quad r = \pm i\sqrt{\lambda} \quad \Rightarrow \quad y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$ where

$y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 \sin\sqrt{\lambda} L$ since $c_1 = 0$. Since we cannot have a trivial

solution, $c_2 \neq 0$ and thus $\sin\sqrt{\lambda}\, L = 0 \quad \Rightarrow \quad \sqrt{\lambda}\, L = n\pi$ where n is an integer $\quad \Rightarrow \quad \lambda = n^2\pi^2/L^2$ and

$y = c_2 \sin(n\pi x/L)$ where n is an integer.

Problem 5

(a) Here you can substitute directly to calculate the Fourier transform, so you have

$$F(k)=\frac{1}{2}[\int_{-\infty}^{+\infty}e^{-i2\pi(k-2k_o)x}dx+\int_{-\infty}^{+\infty}e^{-i2\pi(k+2k_o)x}dx]=\frac{1}{2}\{\delta(k-2k_o)+\delta(k+2k_o)\}$$

(b) Here you can substitute directly to calculate the Fourier transform, so you have

$$F(k)=\frac{1}{4}[\int_{-\infty}^{+\infty}e^{-i2\pi(k-4k_o)x}dx+\int_{-\infty}^{+\infty}e^{-i2\pi(k+4k_o)x}dx+2\int_{-\infty}^{+\infty}e^{-i2\pi kx}dx]=\frac{1}{4}\{\delta(k-4k_o)+\delta(k+4k_o)+2\delta(k)\}$$

Problem 6

(a)

The solution is determined by the separation of variables (the Fourier method):

$$u(x,t) = F(x)G(t).$$

Then

$$\frac{\partial u}{\partial t} = FG', \quad \frac{\partial^2 u}{\partial x^2} = F''G$$

Substituting this into one-dimensional heat equation and separating variables,

$$FG' = c^2 F''G$$

$$\frac{G'}{c^2 G} = \frac{F''}{F} = const = -p^2$$

we obtain the differential equations for $G(t)$ and $F(x)$

$$G' + c^2 p^2 G = 0,$$

$$F'' + p^2 F = 0.$$

Satisfy the boundary conditions:

$$u(0,t) = F(0)G(t) = 0, \qquad u(L,t) = F(L)G(t) = 0, \quad t \geq 0.$$

Thus,

$$F(0) = 0, \qquad F(L) = 0.$$

The general solution for F is

$$F = A\cos px + B\sin px.$$

and

$$F(0) = 0: \qquad A = 0; \qquad F(L) = 0: \qquad B\sin pL = 0$$

which yields

$$\sin pL = 0 \quad (B \neq 0)$$

$$pL = n\pi, \quad p = p_n = \frac{n\pi}{L} \quad (n = 1, 2, \ldots).$$

$$F = F_n = \sin p_n x = \sin \frac{n\pi}{L} x \quad (n = 1, 2, \ldots).$$

The equation for G becomes

$$G' + \lambda_n^2 G = 0, \quad \lambda_n = \frac{cn\pi}{L}.$$

The general solution of this equation is

$$G(t) = G_n(t) = B_n e^{-\lambda_n^2 t} \quad (n = 1, 2, \ldots).$$

Hence the solutions of

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L$$

satisfying

$$u(0, t) = 0, \qquad u(L, t) = 0, \quad t \geq 0.$$

are

$$u_n(x, t) = F_n(x) G_n(t) = B_n e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x \quad (n = 1, 2, \ldots).$$

These functions are called **eigenfunctions** and

$$\lambda_n = \frac{cn\pi}{L}$$

are called **eigenvalues**.

Now we can solve the entire problem by setting

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x.$$

Satisfy the initial conditions:

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x).$$

Thus,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx, \qquad n = 1, 2, \ldots.$$

(b) Satisfy the initial conditions:

$$U(x,0)=20\sin(\pi x/l)+50\sin(3\pi x/L)$$

Only the terms $\lambda_1=c\pi/L$ and $\lambda_3=3c\pi/L$ give non-zero contribution to the series solution with $B_1=20$ and $B_3=50$

So the solution is:

$$U(x,t)=20\exp[-\lambda_1^2 t]\sin(\pi x/L)+50\exp[-\lambda_3^2 t]\sin(3\pi x/L)$$